Accelerated Mathematics III Frameworks Student Edition

Unit 6 Trigonometric Identities, Equations, and Applications

2nd Edition
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Accelerated Mathematics III – Unit 6
Trigonometric Identities
Student Edition

INTRODUCTION:

In previous units, students have defined trigonometric functions using the unit circle, and have also investigated the graphs of the six trigonometric functions. This unit builds on students' understanding of the trigonometric functions by having them discover, derive, and work with some of the most important trigonometric identities. Before applying these identities in problem-solving contexts, it is important that students have a conceptual understanding of their origins. For this reason, most of the identities are derived in the tasks in which they are introduced. These identities are then applied to solve problems in which an identity gives an expression a more convenient form. In unit 7, these identities are revisited in the context of solving trigonometric equations.

ENDURING UNDERSTANDINGS:

- An identity is a statement that is valid for all values of the variable for which the expressions in the equation are defined.
- Trigonometric identities are valuable in a wide variety of contexts because they allow for expressions to be represented in more convenient forms.

KEY STANDARDS ADDRESSED:

MA3A5. Students will establish the identities below and use them to simplify trigonometric expressions and verify equivalence statements.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

MA3A6. Students will solve trigonometric equations both graphically and algebraically.

a. Solve trigonometric equations over a variety of domains, using technology as appropriate.

b. Use the coordinates of a point on the terminal side of an angle to express *x* as $r\cos\theta$ and *y* as $r\sin\theta$.

d. Apply the law of sines and the law of cosines.

MA3A7. Students will verify and apply ½ *ab*sinC to find the area of a triangle.

RELATED STANDARDS ADDRESSED:

MA3A2. Students will use the circle to define the trigonometric functions.

a. Define and understand angles measured in degrees and radians, including but not limited to 0°, 30°, 45°, 60°, 90°, their multiples, and equivalences**.**

b. Understand and apply the six trigonometric functions as functions of general angles in standard position.

c. Find values of trigonometric functions using points on the terminal sides of angles in the standard position.

d. Understand and apply the six trigonometric functions as functions of arc length on the unit circle.

e. Find values of trigonometric functions using the unit circle.

MA3P1. Students will solve problems (using appropriate technology).

a. Build new mathematical knowledge through problem solving.

b. Solve problems that arise in mathematics and in other contexts.

c. Apply and adapt a variety of appropriate strategies to solve problems.

d. Monitor and reflect on the process of mathematical problem solving.

MA3P2. Students will reason and evaluate mathematical arguments.

a. Recognize reasoning and proof as fundamental aspects of mathematics.

b. Make and investigate mathematical conjectures.

c. Develop and evaluate mathematical arguments and proofs.

d. Select and use various types of reasoning and methods of proof.

MA3P3. Students will communicate mathematically.

a. Organize and consolidate their mathematical thinking through communication.

b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.

c. Analyze and evaluate the mathematical thinking and strategies of others.

d. Use the language of mathematics to express mathematical ideas precisely.

MA3P4. Students will make connections among mathematical ideas and to other disciplines.

a. Recognize and use connections among mathematical ideas.

b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

MA3P5. Students will represent mathematics in multiple ways.
a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.

Unit Overview:

The launching task introduces the concept of an identity in a context that students should be familiar with from Accelerated Mathematics I. Before students establish identities and use them to solve problems in the later tasks, it is important that they have a good understanding of what an identity is. Once an understanding of the term identity is established, the task leads students through a geometric derivation of the Pythagorean identities. Students then substitute values into these identities as both a review of basic evaluation of trigonometric functions and also as a final opportunity for students to make the idea of an identity more concrete.

In the second task, students derive the sum identity for the sine function, in the process reviewing some of the geometric topics and ideas about proofs learned in Accelerated Mathematics I. This derivation also provides practice with algebraic manipulation of trigonometric functions that include examples of how applying the Pythagorean identities can often simplify a cumbersome trigonometric expression. Students then apply the sum and difference identities for sine and cosine in the context of evaluating trigonometric functions that are not multiples of 30 or 45 degrees.

In the third task, students investigate how a person's altitude on a Ferris wheel changes as a function of the Ferris wheel's angle of rotation. This activity provides an example of how trigonometric functions model phenomena in our lives in a context that allows students to see that doubling the angle in a trigonometric function does not double the output of the trigonometric function, providing motivation for the double angle identities. Students then derive the double angle identities, which will be applied in later tasks.

In the fourth task, students use numerical, graphical, and algebraic representations to examine trigonometric identities. In the first part of the task, students are given three sets of equations, with each set containing exactly one identity. Students will then use one of three strategies to determine which equation in each set is the identity. Once students have presented their findings from part one, they have a chance to work with each representation in part two.

The final task wraps up the unit by requiring students to use their knowledge from the unit to establish more complex identities. Students often confuse establishing identities with solving equations, because students are so used to solving an equation when given one. It is important to make sure students understand that when asked to establish an identity, they are not solving an equation, and the first part of this task addresses this common misconception. Students then are given three sets of identities to establish, with each set requiring the application of a specified group of the basic identities listed in standard MA3A5.

Vocabulary and formulas:

Identity: an equation that is valid for all values of the variable for which the expressions in the equation are defined.

Reciprocal Identities: $\cos\theta = \frac{1}{\sec\theta}$, $\sin\theta = \frac{1}{\csc\theta}$, $\tan\theta = \frac{1}{\cot\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$

Quotient Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean Identities: $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$

Sum & Difference Identities: $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ and $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Double Angle Identities: $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

Trigonometric Equation: An equation containing one of the basic six trigonometric functions.

Trigonometric Function: A function containing one of the basic six trigonometric functions.

DISCOVERING THE PYTHAGOREAN IDENTITIES LEARNING TASK:

An **identity** is an equation that is valid for all values of the variable for which the expressions in the equation are defined.

You should already be familiar with some identities. For example, in Mathematics I, you learned that the equation $x^2 - y^2 = \ x + y \quad x - y$ is valid for all values of x and y.

1. You will complete the table below by first randomly choosing values for the *x*'s and *y*'s, then evaluating the expressions $x^2 - y^2$ and $\ x + y \quad x - y$. The first row is completed as an example.
   a. Since $x^2 - y^2 = \ x + y \quad x - y$ is an identity, what should be true about the relationship between the numbers in the last two columns of each row?

   b. Complete the table below.

x	y	$x^2 - y^2$	$x + y \quad x - y$

2. An identity is a specific type of equation. Many equations are not identities, however, because an equation is not necessarily true for all values of the involved variables. Of the eight equations that follow, only four are identities. Label the equations that are identities as such and provide a counterexample for the equations that are not identities.

   a. $x - 5 \quad x + 5 \ = x^2 - 25$

   b. $x + 5 \ ^2 = x^2 + 25$

   c. $\sqrt{x^2} = |x|$

   d. $\sqrt{x^4} = x^2$

   e. $\sqrt{x^6} = x^3$

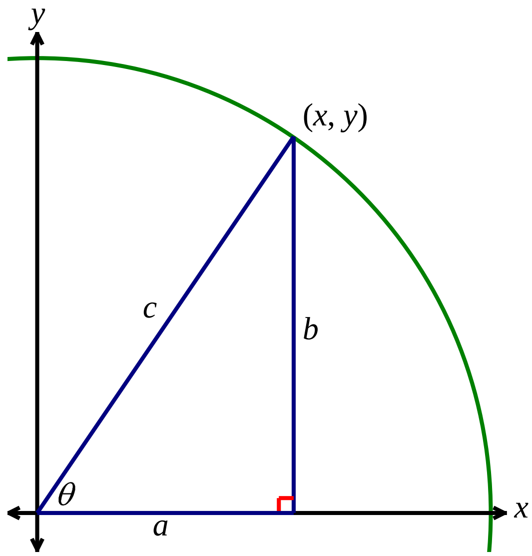
   f. $\sqrt{x^2 + y^2} = x + y$

   g. $a + b \quad a^2 - ab + b^2 \quad = a^3 + b^3$

   h. $\dfrac{x + y}{x} = y$

3. In this unit you will investigate several trigonometric identities. This task looks at the Pythagorean Identities, which are three of the most commonly used trigonometric identities, so-named because they can be established directly from the Pythagorean Theorem.

   In the figure below, the point (x, y) is a point on a circle with radius c. By working with some of the relationships that exist between the quantities in this figure, you will arrive at the first of the Pythagorean Identities



   a. Use the Pythagorean Theorem to write an equation that relates *a, b*, and *c*.

   b. What ratio is equal to $\cos\theta$?

   c. What ratio is equal to $\sin\theta$?

   d. Using substitution and simplification, combine the three equations from parts a-c into a single equation that is only in terms of θ. This equation is the first of the three Pythagorean identities.

4. Since the equation from 3d is an identity, it should be true no matter what θ is. Complete the table below, picking a value for θ that is in the appropriate quadrant. Use your calculator to round values to the nearest hundredth if the angle you choose is not found on the unit circle. How can you use this data to verify that the identity is valid for the four values of θ that you chose?

	θ	$\sin^2\theta$ *	$\cos^2\theta$	$\sin^2\theta+\cos^2\theta$
QI	$\frac{\pi}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	1
QII				
QIII				
QIV				

$*\sin^2\theta = (\sin\theta)^2$, $\cos^2\theta = (\cos\theta)^2$, and so on. This is just a notational convention mathematicians use to avoid writing too many parentheses!

5. The other two Pythagorean identities can be derived directly from the first. In order to make these simplifications, you will need to recall the definitions of the other four trigonometric functions:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

   a. Divide both sides of the first Pythagorean identity by $\cos^2\theta$ and simplify. The result is the second Pythagorean identity.

   b. Divide both sides of the first Pythagorean identity by $\sin^2\theta$ and simplify. The result is the third and final Pythagorean identity.

6. Since the equations from 5a and 5b are identities, they should be true no matter what θ is. Complete the table below, picking a value for θ that is in the appropriate quadrant. Use your calculator to round values to the nearest hundredth if the angle you choose is not found on the unit circle. How can you use this data to verify that identities found in 5a and 5b are both valid for the four values of θ that you chose?

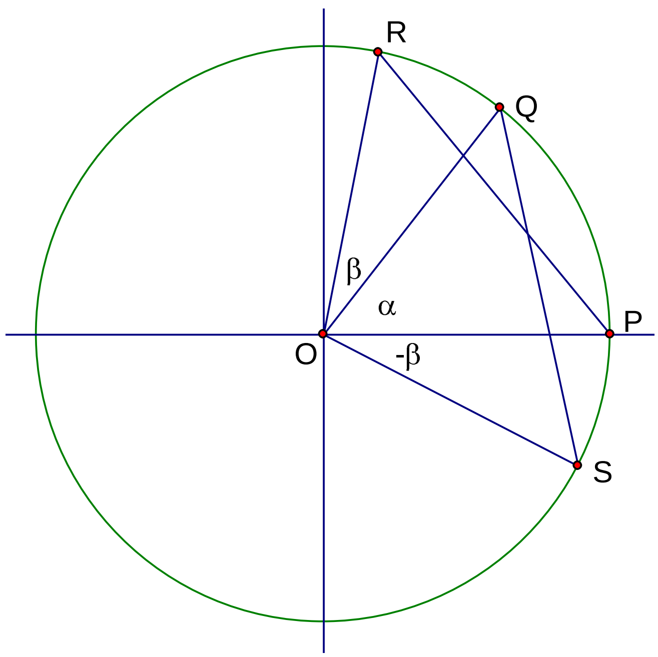
	θ	$1+\tan^2\theta$	$\sec^2\theta$	$1+\cot^2\theta$	$\csc^2\theta$
QI					
QII					
QIII					
QIV					

THE SUM AND DIFFERENCE IDENTITIES LEARNING TASK:

In this task, you will use the sum and difference identities to solve equations and find the exact values of angles that are not multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Before you apply these identities to problems, you will first derive them. The first identity you will prove involves taking the sine of the sum of two angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

We can derive this identity by making deductions from the relationships between the quantities on the unit circle below.



1. Complete the following congruence statements:

a. $\angle ROP \cong \angle QOS$

b. $\overline{RO} \cong \overline{QO} \cong \overline{PO} \cong \overline{SO}$

c. By the ? congruence theorem, $\Delta ROP \cong \Delta QOS$

d. $\overline{RP} \cong \overline{QS}$

2. Write the coordinates of each of the four points on the unit circle, remembering that the cosine and sine functions produce x- and y- values on the unit circle.

a. R = ($\cos\ \alpha+\beta$, $\sin\ \alpha+\beta$)

b. Q = ($\cos\alpha$, $\sin\alpha$)

c. P = (**?,?**)

d. S = ($\cos(-\beta)$, $\sin(-\beta)$)

3. Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{RP}$.

4. **a.** Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{QS}$.

b. Two useful identities that you may choose to explore later are $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Use these two identities to simplify your solution to 4a so that your expression has no negative angles.

5. From 1d, you know that $\overline{RP} \cong \overline{QS}$. You can therefore write an equation by setting the expressions found in problems 3 and 4b equal to one another. Simplify this equation and solve for $\sin(\alpha+\beta)$. Applying one of the Pythagorean Identities will be useful! When finished, you will have derived the angle sum identity for sine.

The other three sum and difference identities can be derived from the identity found in problem 5. These four identities can be summarized with the following two statements.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Recall that so far, you can only calculate the exact values of the sines and cosines of multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. These identities will allow you to calculate the exact value of the sine and cosine of many more angles.

6. Evaluate $\sin 75°$ by applying the angle addition identity for sine and evaluating each trigonometric function:

$$\sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$$

7. Similarly, find the exact value of the following trigonometric expressions:

a. $\cos(15°)$

b. $\sin\left(\frac{\pi}{12}\right)$

c. $\cos(345°)$

d. $\sin\left(\frac{19\pi}{12}\right)$

RIDING ON THE FERRIS WHEEL LEARNING TASK:

1. Lucy is riding a Ferris wheel with a radius of 40 feet. The center of the wheel is 55 feet off of the ground, the wheel is turning counterclockwise, and Lucy is halfway up the Ferris wheel, on her way up. Draw a picture of this situation with Lucy's position and all measurements labeled.

2. If the wheel makes a complete turn every 1.5 minutes, through what angle, in degrees, does the wheel turn each second?

3. Draw a picture showing Lucy's position 10 seconds after passing her position in problem 1? What height is Lucy at in this picture?

4. Draw a picture showing Lucy's position 20 seconds after passing her position in problem 1? What height is Lucy at in this picture?

5. Draw a picture showing Lucy's position 40 seconds after passing her position in problem 1? What height is Lucy at in this picture?

6. Write an expression that gives Lucy's height *t* seconds after passing her position in problem 1, in terms of *t*.

7. In problems 4 and 5, the angle through which Lucy turned was twice that of the problem before it. Did her change in height double as well?

Students commonly think that if an angle doubles, then the sine of the angle will double as well, but as you saw in the previous problems, this is not the case. The double angle identities for sine and cosine describe exactly what happens to these functions as the angle doubles. These identities can be derived directly from the angle sum identities, printed here for your convenience:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

8. Derive the double angle identity for cosine by applying the angle sum identity.

$\cos(2\theta)=\cos(\theta+\theta)=$

9. Derive the double angle identity for sine, also by applying the angle sum identity.

$$\sin(2\theta) = \sin(\theta + \theta) =$$

WHERE'S THE IDENTITY? LEARNING TASK:

Recall that a trigonometric identity is a trigonometric equation that is valid for all values of the variables for which the expression is defined. Sometimes it is very difficult to glance at a trigonometric equation and determine if it is an identity or not. For example, in each of the following sets of three equations, only one is an identity. There are a variety of strategies for determining the identity in each set. Your teacher will assign you to one of the following three strategies for identifying each identity: *Graphical*, *Numerical*, or *Algebraic*. You will then present your strategy to your classmates, listen to your classmates present their strategies, and finally you will practice using each strategy.

Set 1: a. $\sin x \tan x = \sin x$ b. $\cos x \tan x = \sin x$ c. $\sin x \cot x = \sin x$

Set 2: a. $\tan x + \sin x = \dfrac{\sin x + 1}{\cos x}$ b. $\cot x + \csc x = \dfrac{\sin x + 1}{\cos x}$ c. $\tan x + \sec x = \dfrac{\sin x + 1}{\cos x}$

Set 3: a. $\dfrac{\sin x + \cos x}{\sin x \cos x} = \sec x + \csc x$ b. $\dfrac{\sin x \cos x}{\sin x + \cos x} = \sec x + \csc x$ c. $\dfrac{2}{\cos x + \sin x} = \sec x + \csc x$

Graphical

The key to spotting identities graphically is to think of each side of the equation as a function. Since the left side of the equation should always produce the same output as the right side, no matter what the input variable is, both sides should look the same graphically. If the two sides have different graphs, then the equation cannot be an identity.

1. **Set 1:**
   a. In each of the three sets of equations, the right side is the same. Sketch the function made from the right side of the equations in Set 1 in the space provided.

   b. Now sketch each of the three functions made from the left sides of the equations in Set 1. Circle the graph that matches the right side, since that graph is from the identity.
2. **Set 2:**
   a. Sketch the function made from the right side of the equations in Set 2 in the space provided.

   b. Now sketch each of the three functions made from the left sides of the equations in Set 2. Circle the graph that matches the right side, since that graph is from the identity.
3. **Set 3:**
   a. Sketch the function made from the right side of the equations in Set 3 in the space provided.

   b. Now sketch each of the three functions made from the left sides of the equations in Set 3. Circle the graph that matches the right side, since that graph is from the identity.

Numerical

By making a table of values that compares the left side of each equation to the right side, we can rule out equations whose left and right sides do not match. Since we know one of the equations is an identity, the equation that we cannot rule out must be the identity. Complete the following tables, using values of your choosing for x, and use the data you collect to decide which equation in each set is an identity.

1. Set 1:

a. Since the right side of the equation is the same for all three equations in set 1, first complete the following table of values for the right sides of the equations, so that you can compare these values to the left sides of each equation.

x	$\sin x$
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	

b. Now complete each of the three tables of values for the left sides of the equations in Set 1, using the same x-values from part a. Circle the table that matches the right side (from part a), since that table is from the identity.

x	$\sin x \tan x$
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	

x	$\cos x \tan x$
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	

x	$\sin x \cot x$
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	

2. Set 2:

a. Since the right side of the equation is the same for all three equations in set 2, first complete the following table of values for the right sides of the equations, so that you can compare these values to the left sides of each equation.

x	$\sin x$

b. Now complete each of the three tables of values for the left sides of the equations in Set 1. Circle the table that matches the right side (from part a), since that table is from the identity.

x	$\tan x + \sin x$

x	$\cot x + \csc x$

x	$\tan x + \sec x$

3. Set 3:

a. Since the right side of the equation is the same for all three equations in set 3, first complete the following table of values for the right sides of the equations, so that you can compare these values to the left sides of each equation.

x	$\sec x + \csc x$

b. Now complete each of the three tables of values for the left sides of the equations in Set 3. Circle the table that matches the right side (from part a), since that table is from the identity.

x	$\dfrac{\sin x + \cos x}{\sin x \cos x}$

x	$\dfrac{\sin x \cos x}{\sin x + \cos x}$

x	$\dfrac{2}{\cos x + \sin x}$

Algebraic

Recall that from the definitions of the trigonometric functions, we get the following fundamental identities:

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\sin x}{\cos x}$$

Reciprocal Identities

$$\sin x = \frac{1}{\csc x} \qquad \csc x = \frac{1}{\sin x}$$

$$\cos x = \frac{1}{\sec x} \qquad \sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{1}{\cot x} \qquad \cot x = \frac{1}{\tan x}$$

We can use these identities to rewrite trigonometric expressions in different forms. For example, in the following equation, we can rewrite the left side using two of the above identities, eventually making the left side identical to the right side, thus proving that the original equation is an identity.

$$\tan x \cot x = 1$$

$$\frac{\sin x}{\cos x}\frac{\cos x}{\sin x} = 1$$

$$1 = 1$$

In each of the following three sets of equations, there is exactly one identity. Attempt to use the above identities to rewrite one side of each equation so that it matches the other side. Since there is only one identity, this will only be possible for one of the equations in each set. Circle that equation, since it is the identity.

Set 1:

$$\sin x \tan x = \sin x \qquad \cos x \tan x = \sin x \qquad \sin x \cot x = \sin x$$

Set 2:

$$\tan x + \sin x = \frac{\sin x + 1}{\cos x} \qquad \cot x + \csc x = \frac{\sin x + 1}{\cos x} \qquad \tan x + \sec x = \frac{\sin x + 1}{\cos x}$$

Set 3:

$$\frac{\sin x + \cos x}{\sin x \cos x} = \sec x + \csc x \qquad \frac{\sin x \cos x}{\sin x + \cos x} = \sec x + \csc x \qquad \frac{2}{\cos x + \sin x} = \sec x + \csc x$$

Using All Three Approaches

Set 4: Graphical

a. $3 = 3\sin^2 x + \cos^2 x$ b. $1 + 2\sin^2 x = 3\sin^2 x + \cos^2 x$ c. $3\sin^2 x = 3\sin^2 x + \cos^2 x$

a. In each of the three sets of equations, the right side is the same. Sketch the function made from the right side of the equations in Set 4 in the space provided, with the aid of a graphing utility.

b. Now sketch each of the three functions made from the left sides of the equations in Set 1. Circle the graph that matches the right side, since that graph is from the identity.

Set 5: Numerical

a. $2\sin x = \tan x \cos x + \sin x$ b. $\tan x = \tan x \cos x + \sin x$ c. $\cos x \sin x = \tan x \cos x + \sin x$

a. Since the right side of the equation is the same for all three equations in set 1, first complete the following table of values for the right sides of the equations, so that you can compare these values to the left sides of each equation.

x	$\tan x \cos x + \sin x$

b. Now complete each of the three tables of values for the left sides of the equations in Set 1, using the same x-values from part a. Circle the table that matches the right side (from part a), since that table is from the identity.

x	$2\sin x$

x	$\tan x$

x	$\cos x \sin x$

Set 6: Algebraic

Attempt to use the reciprocal and quotient identities to rewrite one side of each equation so that it matches the other side. Since there is only one identity, this will only be possible for one of the equations. Circle that equation, since it is the identity.

$$\frac{\tan x + \sec x}{\cos x} = \sin x + 1 \qquad \frac{\cot x + \csc x}{\csc x} = \sin x + 1 \qquad \frac{\tan x + \sec x}{\sec x} = \sin x + 1$$

ESTABLISHING IDENTITIES LEARNING TASK:

As you saw in *Where's the Identity?*, trigonometric identities can be difficult to recognize, but by thinking graphically, numerically, and algebraically, you can gain valuable insight as to whether equations are identities or not. Numerical and graphical information is not enough to verify an identity, however.

Identities can be established algebraically by rewriting one side of the equation until it matches the expression on the other side of the equation. Rewriting is often done by applying a basic trigonometric identity, so the identities you have already established are listed here for reference.

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta \qquad \tan^2\theta = \sec^2\theta - 1 \qquad 1 = \csc^2\theta - \cot^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta \qquad 1 = \sec^2\theta - \tan^2\theta \qquad \cot^2\theta = \csc^2\theta - 1$$

Sum & Difference Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

When establishing identities, it is important that each equation that you write is logically equivalent to the equation that precedes it. One way to ensure that all of your equations are equivalent is to work with each side of the equation independently. The following problem provides an example of how failing to work with each side of an equation independently can produce what appears to be a proof of a statement that isn't true. This example should serve as a reminder as to why you should work with each side of an equation independently when establishing identities.

1. As explained above, a helpful guideline when establishing identities is to change each side of the equation independently. Circle the two lines that were produced by failing to abide by this guideline, in the "faulty proof" below.

$$\sin\theta = -\sqrt{1-\cos^2\theta}$$

2. Explain why $\sin\theta = -\sqrt{1-\cos^2\theta}$ is not an identity, using either graphical or numerical reasoning.

When establishing the following identities, keep the following two general rules of thumb in mind. They will not always lead to the most efficient solution, but they are usually beneficial when help is needed.

- Begin working on the most complex side, because it is usually easier to simplify an expression rather than make it more complex.
- When no other solution presents itself, rewrite both sides in terms of sines and cosines.

Establish the following identities by rewriting the left, right, or both sides of the equation independently, until both sides are identical.

For problems 3-5, apply either the quotient or reciprocal identities.

3. $\sin x \cot x = \cos x$

4. $\csc\theta \tan\theta \cos\theta = 1$

5. $\dfrac{\sec\alpha}{\csc\alpha} = \tan\alpha$

For problems 6-10, apply the Pythagorean Identities.

6. $\tan^2 w + 8 = \sec^2 w + 7$

7. $4\cos^2\theta + 3\sin^2\theta = 3 + \cos^2\theta$

8. $\dfrac{\sin\theta \tan\theta}{1-\cos\theta} = \sec\theta + 1$

9. $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

For problems 10-12, apply the sum, difference, or double angle identities.

10. $\dfrac{\cos\ \ \alpha+\beta}{\cos\alpha \sin\beta} = \cot\beta - \tan\alpha$

11. $\dfrac{\sin\ \ \alpha-\beta}{\sin\alpha \cos\beta} = 1 - \cot\alpha \tan\beta$

12. $2\cot x \cot\ \ 2x\ \ = \cot^2 x - 1$

13. $\cos^4\theta - \sin^4\theta = \cos\ \ 2\theta$

Solving Equations Learning Task:

As you have just seen with identities, two trigonometric expressions can always, sometimes, or never be equal. Identities occurred when the two expressions were always equal. Now, in this task, we consider the sometimes and never situations. We will continue to look at these situations algebraically, graphically, and numerically.

Consider this situation:

The rabbit population in a national park rises and falls each year, it reaches its minimum of 5000 rabbits in January. By July, as the weather warms up and food grows more abundant, the population triples in size. By the following January, the population again falls to 5000 rabbits, completing the annual cycle. Using the given data for the two-year cycle of rabbit population, write a formula for the number of rabbits, R, as a function of time in months, for $0 \leq t \leq 24$.

t	R
0	5000
6	15000
12	5000
18	15000
24	5000

1. What is the midline for this function?

2. When is the rabbit population at the midline value?

3. How do you find all the values of t when R is the midline value? Algebraically? Graphically? Using a table? Why did you pick the method you picked?

4. When is the rabbit population 12,000? How did you find these values?

5. When is the rabbit population more than 12,000? How do you know?

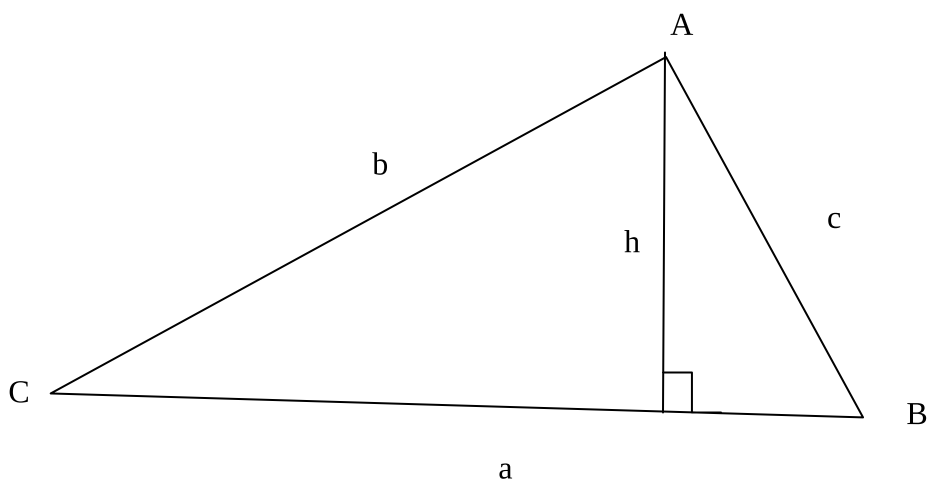
6. When is the rabbit population 4000? How do you know?

The NON-Right Triangle Learning Task:

The sine cosine and tangent functions are useful when solving within a right triangle situation. However, not all situations can be defined as right triangles. Therefore, we need other relationships between sides and angles of non-right triangles to solve any problem situation we may be presented with. In this task, you will do so by establishing that result in the Law of Sines and the Law of Cosines.

The Law of Sines

Given triangle ABC with altitude from A, consider the following relationship:



Sin C = ?   Sin B = ?

Solve each equation for h.

If both equations now equal h, they are equal to each other, when you solve and re-write, you should have: $\frac{Sin\,B}{b} = \frac{Sin\,C}{c}$, you can solve a similar situation with side a and angle A which leads to the extended proportion known as the Law of Sines:

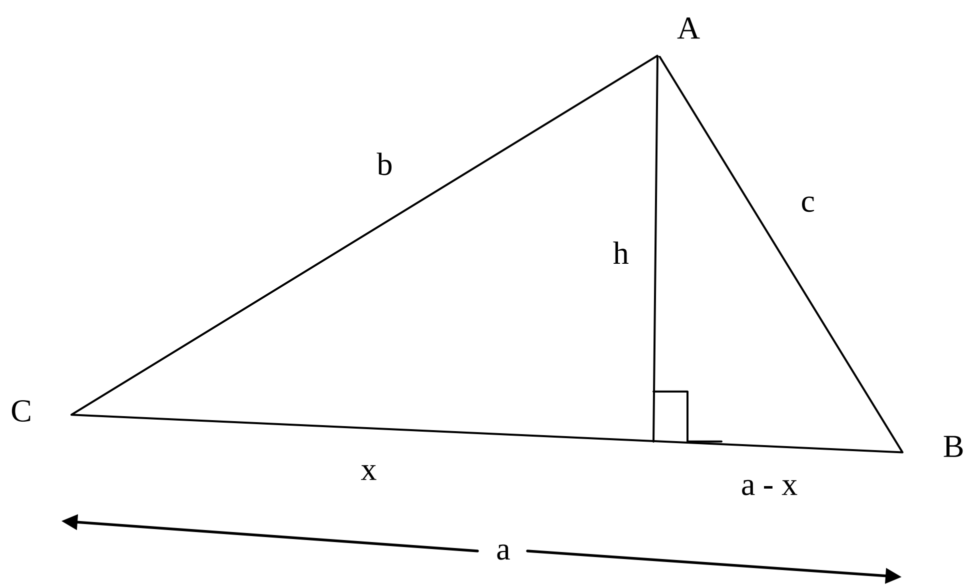
$$\frac{Sin\,A}{a} = \frac{Sin\,B}{b} = \frac{Sin\,C}{c}$$

The Law of Sines is useful when sides and opposite angles are given. Think back to the triangle congruence theorems you learned in Accelerated Mathematics I, which of those apply here?

Suppose an aerial tram starts at a point one half mile from the base of a mountain whose face has a 60^o angle of elevation. The tram ascends at an angle of 20^o. What is the length of the cable needed to span the distance from T to A?

The Law of Cosines

Given triangle ABC with altitude from A, consider the following relationship:



Since there is a right angle where the altitude intersects side BC, both of the triangles formed are right triangles. That means that the sides must satisfy the Pythagorean Theorem.

Write the appropriate Pythagorean statement for the right triangle with sides c, h, and a – x.

Expand the squared binomial.

But, we want a relationship between the sides and the angles of triangle ABC, this means we need to get rid of the x and h. What do we know about the relationship between x and h. Use that relationship to replace them in your equation.

We have almost gotten the equation in terms of the sides and angles of triangle ABC, but now we must find a replacement for x. What relationships can you find with the x, h, c, triangle that might help? How would you substitute that relationship into your equation?

Thus, the Law of Cosines is $a^2 + b^2 – 2ab\text{Cos } C = c^2$ for the sides a, b, c and opposite angle C for any triangle.

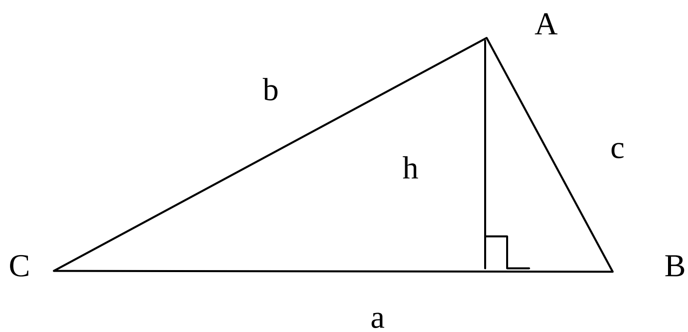
The Law of Cosines is useful when given two sides and the included angle or when given three sides of a triangle. Think back to the triangle congruence theorems you learned in Accelerated Mathematics I, which of those apply here?

For background information that may be helpful with this task visit:
http://volcanoes.usgs.gov/hvo/activity/maunaloastatus.php

To help predict eruptions from the volcano Mauna Loa on the island of Hawaii, scientists keep track of the volcano's movement by using a "super triangle" with vertices on the three volcanoes Mauna Loa, Mauna Kea, and Hualalai. For example, in recent year, Mauna Loa moved 6 inches as a result of increasing internal pressure. If the distance from Mauna Loa to Mauna Kea is 22.479 miles and the distance from Mauna Kea to Haulalai is 28.143 miles and the angle at Mauna Kea is 58. 569^o, how far is it from Mauna Loa to Haulalai? (GPS receivers are use to constantly monitor these line lengths)

The Area of Any Triangle

Given triangle ABC with altitude from A, consider the following relationship:



Using the same idea as we have for the Law of Sines and Cosines, we will now investigate the area of any triangle.

What is the area of the above triangle?

Using what you know about trigonometry, how could you find a value to substitute for h?

Now, make your substitution into the area formula.

Can you generalize the formula in words?

A real estate salesperson wants to find the area of a triangular lot. The surveyor takes measurements for her. He finds that the two sides are 74.3 m and 34.6 m, and the angle between them is 40.2^{o}. What is the area of the lot?

The distance from a point to a line

The final application of our trigonometry is an alternate method from the one you developed in Accelerated Mathematics I of finding the distance between a point and a line.
Explore this relationship using this website:

http://www.mathopenref.com/coordpointdisttrig.html